ON THE PERMANENCE OF EQUIVALENCE.

[From the Discussions of the Mathematical Club, Cornell University.]

The so-called principle of the Permanence of Equivalent Forms was stated by a member as follows: If two symbolic expressions be equivalent when the symbols are general in form but restricted to a certain class of values, they will continue to be equivalent when the symbols are wholly unrestricted in value as well as in form.

It was pointed out that the principle as thus stated might better be called the Permanence of the Equivalence of Forms; and that the author* who gives it the greatest prominence uses the principle chiefly as a basis for interpreting new symbols,—as when the forms $a^m \times a^n$ and a^{m+n} are made permanently equivalent by convention, and thence is derived the interpretation of the symbol a^m when m is fractional or negative; and that he professes to apply the principle only to cases in which the fundamental definitions of the symbols cease to be applicable, thus disclaiming its validity in removing the restrictions from such a formula as

$$\sin(\theta + \theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta',$$

supposed to be proven for positive acute angles.

It was noticed, however, that Peacock himself in proving the general Binomial Theorem by Euler's method uses the principle in question to demonstrate the permanence of the fundamental equation

$$f(m) \times f(n) = f(m + n),$$

wherein $f(m)$ stands for the "binomial expansion"

$$1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots,$$

thus pushing the principle beyond the field of convention into that of logic, since here no question of interpreting new symbols is involved; and that this illustrates the need of some demonstrated and more sharply-defined principle applicable to such cases.

This need is likewise suggested by the fact that Euler disclaims the universality of the principle of the Permanence of Equivalence, citing as an instance of its failure the equation

$$m = \frac{1-a^m}{1-a} + \frac{(1-a^m)(1-a^{m-1})}{1-a^2} + \frac{(1-a^m)(1-a^{m-1})(1-a^{m-2})}{1-a^3} + \ldots,$$

* Peacock's Symbolic Algebra.

which is true only when m is an integer; so that Peacock charges him with denying the generality of the only principle that could give validity to his proof of the Binomial Theorem.

It was also remarked that although the principle of permanence is most plausible when applied to ordinary algebraic polynomials, yet this plausibility is weakened in the present case by the fact that the expressions $f(m), f(n)$, $f(m+n)$, taken as wholes, change their character in one important particular when m, n take their unrestricted values: viz., that $f(m)$, which consists of a finite number of terms when m is a positive integer, becomes an infinite series in all other cases, and that moreover in certain cases $f(m)$ and $f(n)$ might both be infinite series, and yet $f(m+n)$ be a finite series.

In view of the wide-spread impression that Euler's proof of the Binomial Theorem is wanting in logical rigor, some of the members had sought for an algebraic principle that should fill out what was probably in Euler's mind. The purpose was served by the "Principle of Algebraic Identity," which is a special case of the permanence of equivalence, and which it is convenient here to enunciate separately for the cases of one and of two variables:—

(a) If there be two rational integral functions involving m in degrees not exceeding r, and if these functions be equal for more than r separate values of m,—then they are equal for all values of m; i. e., they are identical and the coefficients of corresponding terms are equal.

(b) If there be two rational integral functions, each involving m in degrees not exceeding r, and n in degrees not exceeding s, and if there be certain definite values of n, more than s in number, such that when n has any one of these values the functions are equal for more than r separate values of m,—then the functions are equal for all values of m and n, and the coefficients of corresponding terms are equal.

The first of these principles is derived, in most works on algebra, from the theorem that a function of the rth degree cannot vanish for more than r separate values of its variable. It can be extended without difficulty to functions of two variables as enunciated in the second principle, and also to functions of more than two variables if desired.

These principles were applied to establish the permanence of the equivalence in question as follows: Suppose the series $f(m)$ to be actually multiplied by $f(n)$, and the product, arranged in ascending powers of x, to be compared with the series $f(m+n)$ similarly arranged. It appears by inspection that the first two or three terms correspond; and it remains to be shown that the general terms also correspond. For let the coefficients of x^r in $f(m) \times f(n)$ and in $f(m+n)$, respectively, be denoted by $\varphi_r(m, n)$, $\theta_r(m, n)$; then the functions φ_r, θ_r involve m and n in degrees not exceeding r, and are numerically

equal for every positive integral value of m and n; hence, by the second principle, $\varphi_r(m, n)$ and $\theta_r(m, n)$ are equal for all values of m and n, and are algebraically identical. But these represent any two corresponding coefficients in the two series in question, hence the series $f(m) \times f(n)$ and $f(m+n)$ are identical term by term.

Thus the permanence of Euler's fundamental equivalence was established, from which the proof of the general Binomial Theorem easily follows.*

It was pointed out that Euler's is to be regarded as the most philosophical proof of the Binomial Theorem, growing as it does by a natural and general method out of the result for a positive integer, which itself rests naturally on the theory of combinations.

Brief reference was also made to a further extension of the same theorem, in which the terms of the binomial are replaced by general functional symbols of operation, the exponents indicating repetitions of these operations; and it was shown that the extension is valid only when the symbols follow the same commutative, associative, and distributive laws that lie at the basis of multiplication; these laws are fulfilled by imaginary numbers regarded as versi-tensors,† and by many operators in the method of differences, finite and infinitesimal; e. g. the operator $\left[a\dfrac{d}{dx} + b\dfrac{d}{dy}\right]^n$, acting on any function of x and y, is equivalent to the operator

$$a^n \frac{d^n}{dx^n} + na^{n-1}b\frac{d^n}{dx^{n-1}dy} + \frac{n(n-1)}{1.2}a^{n-2}b^2\frac{d^n}{dx^{n-2}dy^2} + \ldots .$$

In such cases, where there is no interpretation of new symbols, the validity of the result does not depend on the general principle of permanence, but on some special statement of it which determines what particular properties of the restricted symbols are to be preserved in the more general ones.

On the other hand, where there are new symbols to be interpreted, it is usual to fix on some controlling equivalence or equivalences which are to remain permanent, and in accordance with which the new symbols are to be assigned their meaning. Of this process the two following examples were given:—

(1) The interpretation of x^m, when m is imaginary, is obtained by making permanent the exponential expansion

$$x^m = 1 + \lambda m + \frac{\lambda^2 m^2}{1.2} + \ldots, \qquad [\lambda = \log_e x$$

whence x^{a+bi} is to be equivalent to the operator $x^a(\cos\lambda b + i \sin\lambda b)$, whose tensor is x^a, and versorial angle b radians; and the more general symbol

* See Hall and Knight's, or Todhunter's Algebra.

† Oliver, Wait, and Jones's Algebra, p. 264 *et seq.*

$(x+yi)^{a+bi}$, to the operator whose tensor is $r^a e^{-b\theta}$, and versorial angle $a\theta + b\lambda$, where r and θ are the tensor and versorial angle of $x + yi$, and $\lambda = \log_e r$. This interpretation preserves the "laws of exponents," and includes the ordinary meanings as special cases. The Binomial Theorem may then be correspondingly extended by establishing the algebraic identity of the two series

$$1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots,$$

$$1 + \lambda m + \frac{\lambda^2 m^2}{1.2} + \ldots,$$

where $\lambda = \log_e(1+x) = x - \frac{x^2}{2} + \ldots$, the two series being known to be equivalent for all real values of x and m that make them both convergent.

(2) An interpretation of $\left[\frac{d}{dx}\right]^m$ when m is fractional or imaginary,* may be derived by means of the equivalence

$$\left[\frac{d}{dx}\right] \cdot \left[\frac{d}{dx}\right]^n . \, u = \left[\frac{d}{dx}\right]^{m+n} . \, u,$$

in conjunction with one of the auxiliary equations

$$\left[\frac{d}{dx}\right]^r . \, e^{ax} = a^r . \, e^{ax},$$

$$\left[\frac{d}{dx}\right]^r . \, x^p = \frac{\Gamma(p+1)}{\Gamma(p-r+1)} x^{p-r},$$

in which the "gamma function" $\Gamma(p+1)$ is identical with $p!$ when p is a positive integer.

The whole discussion showed that the general principle of permanence should be limited to the field of interpretation; and that beyond that field we need to give a precise and special statement to it, and to the nature of the restrictions whose unimportance it asserts; such special statement requiring demonstration from the properties of the symbols involved.

Another aspect of this question will be treated of in connection with the "Principle of Continuity," which will be the subject of a future discussion.

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* See De Morgan's Calculus, p. 599, with references; Kelland on "General Differentiation," in Edinburgh Trans., Vol. XIV, with interesting applications; also Centre's papers in Cambridge and Dublin Math. Journal, Vols. III, IV, V.